Solving Differential Equations
Difficulty: Easy

Model Answers 1

Time allowed: **32 minutes**

Score: **/27**

Percentage: **/100**

Grade Boundaries:

A*	A	B	C	D	E	U
>76%	61%	52%	42%	33%	23%	<23%

Question 1

(a) Find $\int \frac{9x+6}{x} \, dx$, $x > 0$. **(2)**

$$I = \int \frac{9x+6}{x} \, dx$$

The integrand is easily separable:

$$I = \int \left(9 + \frac{6}{x}\right) dx$$

$$\mathbf{I = 9x + 6\ln(x) + c}$$

(b) Given that $y = 8$ at $x = 1$, solve the differential equation

$$\frac{dy}{dx} = \frac{(9x+6)\,y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2 = g(x)$.

(6)

Separate the variables:

$$\frac{dy}{y^{\frac{1}{3}}} = \frac{9x+6}{x}\,dx$$

Integrate both sides:

$$\int \frac{dy}{y^{\frac{1}{3}}} = \int \frac{9x+6}{x}\,dx$$

$$\int y^{-\frac{1}{3}}\,dy = 9x + 6\ln(x) + c$$

$$\frac{3y^{\frac{2}{3}}}{2} = 9x + 6\ln(x) + c$$

Substitute $x = 1$ and $y = 8$ into the differential equation to find the constant, c :

$$\frac{3}{2}(8)^{\frac{2}{3}} = 9(1) + 6\ln(1) + c$$

$$6 = 9 + c$$

$$c = -3$$

Rearrange to find the equation in the form "$y^2 =$ ":

$$\frac{3}{2}y^{\frac{2}{3}} = 9x + 6\ln(x) - 3$$

Multiply both sides by $\frac{2}{3}$:

$$y^{\frac{2}{3}} = 6x + 4\ln(x) - 2$$

Cube both sides of the equation:

$$y^2 = (6x + 4\ln(x) - 2)^3$$

$$\boldsymbol{y^2 = 8(3x + 2\ln(x) - 1)^3}$$

(Total 8 marks)

Question 2

(a) Find $\int (4y+3)^{-\frac{1}{2}} \, \mathrm{d}y$

(2)

$$\int (4y+3)^{-\frac{1}{2}} \, dy$$

We can integrate this directly. We can always check this by differentiating our answer, and checking that it is the same expression that we started with inside the integral:

$$= \frac{1}{2}(4y+3)^{\frac{1}{2}} + A$$

(b) Given that $y = 1.5$ at $x = -2$, solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{(4y+3)}}{x^2}$$

giving your answer in the form $y = f(x)$.

(6)

$$\frac{dy}{dx} = \frac{\sqrt{4y+3}}{x^2}$$

We can solve this by separating out variable:

$$\int (4y+3)^{-\frac{1}{2}}\, dy = \int x^{-2}\, dx$$

$$\frac{1}{2}(4y+3)^{\frac{1}{2}} = -\frac{1}{x} + B$$

Use the point $(-2,1.5)$ to find out the value for B

$$\frac{3}{2} = \frac{1}{2} + B$$

$$B = 1$$

$$\frac{1}{2}(4y+3)^{\frac{1}{2}} = -\frac{1}{x} + 1$$

$$4y + 3 = \left(2 - \frac{2}{x}\right)^2$$

$$\mathbf{y = \frac{1}{4}\left(2 - \frac{2}{x}\right)^2 - \frac{3}{4}}$$

(Total 8 marks)

Question 3

Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2 x}$$

(5)

$$\frac{dy}{dx} = \frac{3}{y\cos^2(x)}$$

Using separation of variables we can get the following.

$$\int y\,dy = \int 3\sec^2(x)dx$$

$\int 3\sec^2(x)dx$=tan(x) is a standard integral that you need to know.

$$\frac{y^2}{2} = 3\tan(x) + A$$

Using constraints $\left(\frac{\pi}{4}, 2\right)$:

$$2 = 3 + A$$

$$A = -1$$

Substituting this back into the equation for y.

$$\mathbf{\frac{y^2}{2} = 3\tan(x) - 1}$$

(Total 5 marks)

Question 4

Given that $y = 2$ at $x = \frac{\pi}{8}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3y^2}{2\sin^2 2x}$$

giving your answer in the form $y = f(x)$.

(6)

Separate the variables

$$\frac{1}{3y^2}dy = \frac{dx}{2\sin^2 2x}$$

Integrate

$$\frac{1}{3}\int y^{-2}dy = \frac{1}{2}\int \csc^2 2x\, dx$$

$$\rightarrow -\frac{1}{3}y^{-1} = \frac{1}{2}\left(-\frac{\cot 2x}{2}\right) + c$$

Note that $\frac{d}{dx}(-\cot x) = \csc^2 x$

$$\rightarrow \frac{1}{y} = \frac{3}{4}\cot 2x + c$$

Plug in known values to find *c*

$$\frac{1}{2} = \frac{3}{4}\cot\frac{\pi}{4} + c$$

$$\rightarrow c = \frac{1}{2} - \frac{3}{4}$$

$$\rightarrow c = -\frac{1}{4}$$

Hence

$$\frac{1}{y} = \frac{3}{4}\cot 2x - \frac{1}{4}$$

$$\rightarrow \boldsymbol{y = \frac{4}{3\cot 2x - 1}}$$

(Total 6 marks)